

Single Proxy Identifiability of Causal Effects

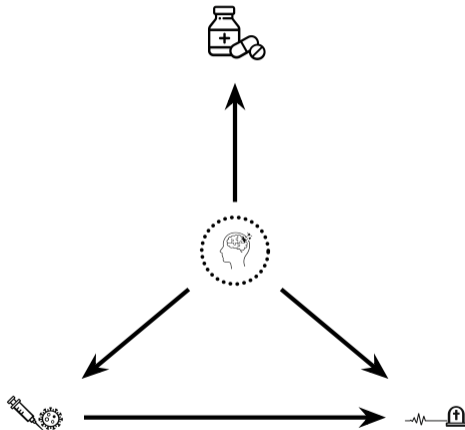
under unobserved confounding and
a known error mechanism

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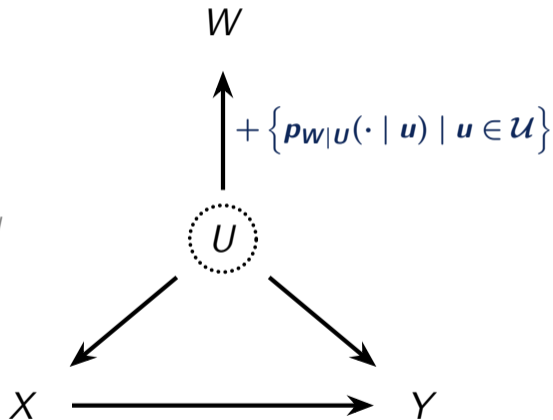
Sebastian Weichwald, Niklas Pfister

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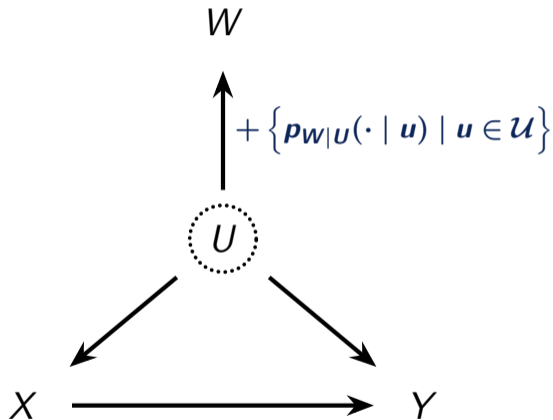


Jackson et al. (2005); Wyss et al. (2022)

treatment $X \in \mathcal{X} \subseteq \mathbb{R}^p$
outcome $Y \in \mathcal{Y} \subseteq \mathbb{R}$
confounder $U \in \mathcal{U} \subseteq \mathbb{R}^k$
proxy $W \in \mathcal{W} \subseteq \mathbb{R}^d$
 $p, k, d \in \mathbb{N}$



We identify
causal effects
under **unobserved**
confounding
and a **known**
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differential privacy



U.S. Census Bureau (2021)

1. Model and causal function

2. Identifiability

3. Method

4. Application

We assume a **Proxy-Confounded Structural Causal Model (PC-SCM)**.

$$M : \begin{cases} U \leftarrow N_U \\ W \leftarrow f_W(U, E) \\ X \leftarrow f_X(U, N_X) \\ Y \leftarrow f_Y(U, X, N_Y) \end{cases}$$

N_U, E, N_X, N_Y jointly independent

$$\implies W \perp\!\!\!\perp (X, Y) \mid U$$

- joint distribution P^M has a density p^M
- interventional distribution $P_Y^{M; do(X:=x)}$ for each $do(X = x)$
- observe n i.i.d. copies of (X, Y, W) generated by M_0
- error mechanism $\{p_{W|U}^{M_0}(\cdot | u) \mid u \in \mathcal{U}\}$ is known

1. Causal function



Definition (causal function)

The **causal function** $\theta^{M_0} : \mathcal{X} \rightarrow \mathbb{R}^q$ is defined for all $x \in \mathcal{X}$ as

$$\theta^{M_0}(x) := \mathbb{E}_Y^{M_0; do(X:=x)} [Y].$$

- binary treatment $X \in \{0, 1\}$:

$$\text{ATE} = \theta^{M_0}(1) - \theta^{M_0}(0)$$

- Adjusting for the measurement W does not suffice.



2. Identifiability



The causal function is identifiable if it is the same across all PC-SCMs that induce the same observational features.



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- for all observationally equivalent PC-SCMs $M \in \mathcal{M}(M_0)$, it holds that $\mathcal{F}^M = \mathcal{F}^{M_0}$



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Definition (identifiability)

θ^{M_0} is **identifiable** if for all models $M \in \mathcal{M}(M_0)$, it holds that

$$\theta^M \equiv \theta^{M_0}.$$

2. Identifiability



The causal function is identifiable if the error mechanism is **L_∞ -complete**, that is, for all $\delta \in L_\infty(\mathcal{U})$

$$\int_{\mathcal{U}} p_{W|U}^{M_0}(\cdot | u) \delta(u) \mu(du) \equiv 0 \quad \implies \quad \delta \equiv 0 \quad \text{a.e.}$$

and for all $M \in \mathcal{M}(M_0)$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$ it holds that $p_{U|X,Y}^M(\cdot | x, y) \in L_\infty$.

Theorem (continuous)

Assume that the SPICE Assumption holds.

Then, the error mechanism $\{p_{W|U}^{M_0}(\cdot | u) \mid u \in \mathbb{R}^k\}$ is L_∞ -complete.

2. Continuous confounders and proxies



Assumption (SPICE)

- (i) (**sufficient dimension**) The proxy $W \in \mathcal{W} \subseteq \mathbb{R}^d$ has at least the dimension of the confounder $U \in \mathcal{U} \subseteq \mathbb{R}^k$, that is, $d \geq k$ and
- (ii) (**additive noise**) for a matrix $A \in \mathbb{R}^{d \times k}$ with full column rank and a random variable $E \in \mathcal{E} \subseteq \mathbb{R}^d$, we have

$$W = AU + E \quad \text{with} \quad U \perp\!\!\!\perp E$$

and

- (iii) (**density**) the distribution of E has a density $p_E^{M_0}$ on \mathbb{R}^d and
- (iv) (**non-vanishing Fourier**) the Fourier transform of $p_E^{M_0}$ has no zeros.

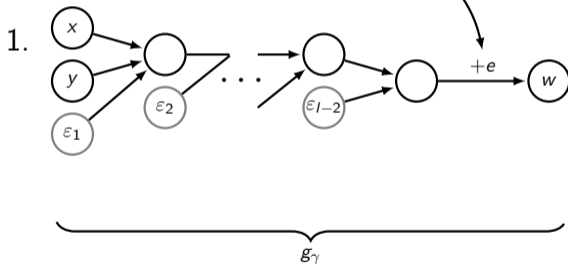


3. Method: SPICE-Net



we know for all $(u, w) \in \mathcal{U} \times \mathcal{W}$ that

$$p_{W|U}^{M_0}(w | u) = p_E^{M_0}(w - Au)$$



optimise weights γ by minimising energy loss Shen and Meinshausen (2025)

2. causal function estimator

guarantees

- $g_\gamma^* - E \sim P_{\tilde{A}U|X,Y}^{M_0}(x,y)$
for almost all $(x, y) \in \mathcal{X} \times \mathcal{Y}$
- adjusting for $\tilde{A}U$ is fine

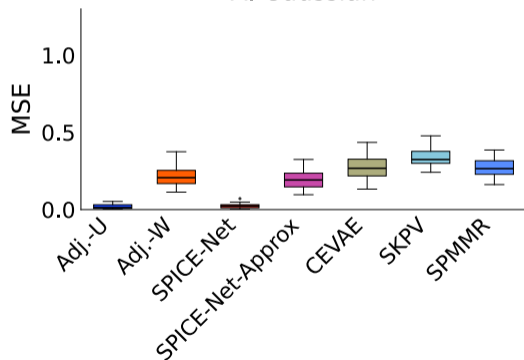


4. Simulation



Gaussian

A. Gaussian



$$U \sim \mathcal{N}(0, 1)$$

$$f_W \sim U + \mathcal{N}(0, 1)$$

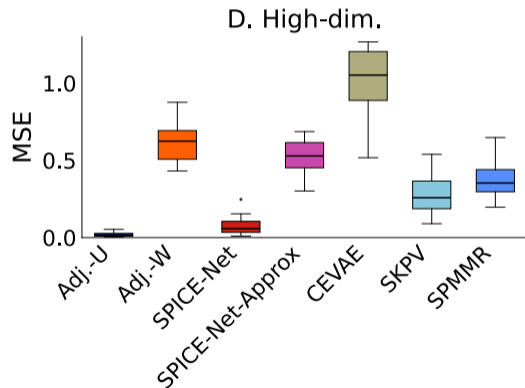
$$f_X \sim U + \mathcal{N}(0, 1)$$

$$f_Y \sim U + X + \mathcal{N}(0, 1)$$

4. Simulation



High-dimensional



$$U \sim \mathcal{N}_2(0, I)$$

$$f_W = AU + E$$

$$f_X = BU + \mathcal{N}(0, 1)$$

$$f_Y = BU + X + \mathcal{N}(0, 1)$$

with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

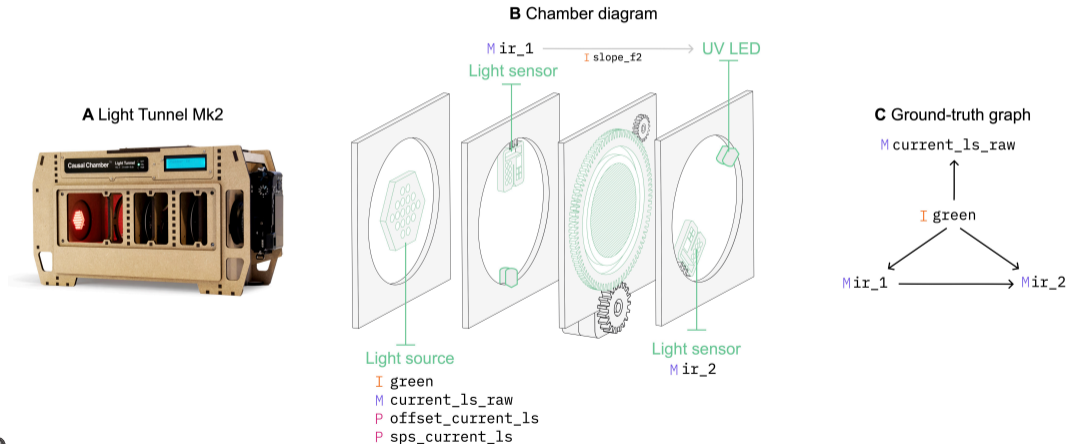
$$E \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2.0 & -0.3 & 0.5 \\ -0.3 & 1.5 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix} \right)$$



4. Real-world data



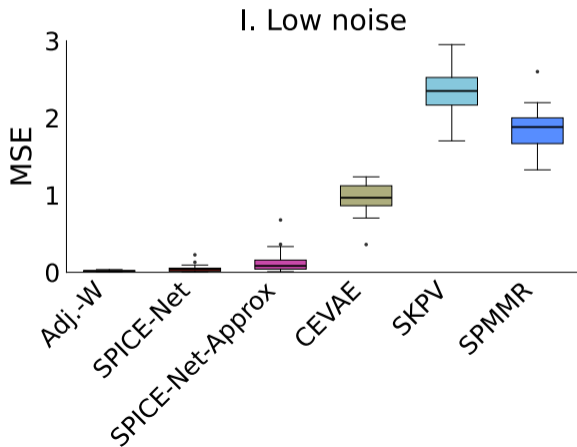
Causal Chamber[®] Gamella et al. (2025)



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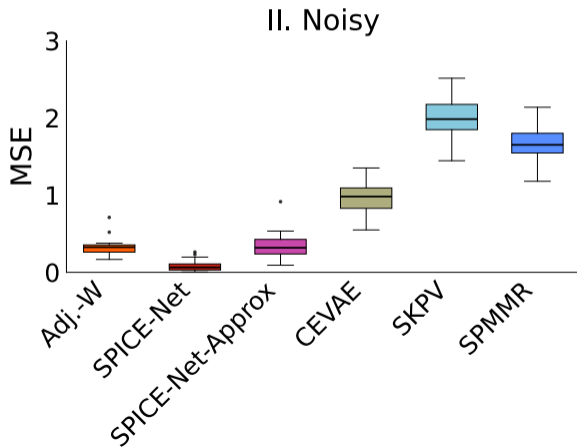
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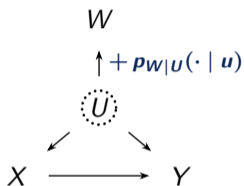
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Identifying Causal Effects Using a Single Proxy Variable



1. Model



identifiability

$$\mathcal{F}^M = \mathcal{F}^{M_0} \Rightarrow \theta^M \equiv \theta^{M_0}$$

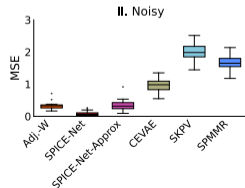
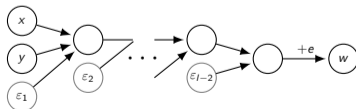


2. SPICE

The error mechanism is complete if

- sufficient proxy dimension $d \geq k$
- $W = AU + E$ with $U \perp\!\!\!\perp E$
- non-vanishing Fourier of $p_E^{M_0}$

3. SPICE-Net



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